The Chebychev best approximation for unsolvable systems of Max-Min fuzzy equations

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Abstract For unsolvable systems of linear equations of the form \(A \otimes x = b\) over the max-min algebra \(R = ([0, 1], \oplus = \max, \otimes = \min)\), we propose an efficient method for finding a Chebychev-best approximation \(\hat{b} \in R(m)\) of the vector \(b \in R(m)\) in the set \(S(A) = \{\hat{b} \in R(m); A \otimes x = \hat{b} \text{ is solvable}\}\). Moreover an algorithm for testing the method is reviewed.

1 Introduction

In many applications, e.g. in fuzzy control systems, in discrete dynamic systems, or in knowledge engineering, fuzzy relation equations play important role. Application of systems of linear equations over structures different from the classical field were studied in the 1960s and in recent years have again received increasing attention as a tool for modeling discrete event systems or fuzzy relations. Kim and Roush [7] generalized the notion in various aspects, as for instance fuzzy matrices over the two element Boolean algebra, over the nonnegative real numbers, over the nonnegative integers, and over other semi-rings. Several authors observed that a system of the form \(A \otimes x = b\) always has a 'principal' solution \(x^*\), i.e. such that the system is solvable if and only if \(x^*\) is a solution, and in this case \(x^*\) is the maximum solution [5]. The solvability of fuzzy equations in G"odel algebra was first studied in [10], and then Cechl"arová [1] discussed unique solvability and strong regularity of matrices, and also formulated a necessary and sufficient condition for a linear system of equations

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over a fuzzy algebra to have a unique solution. Ragab and Emam [9] focused on square fuzzy matrices, their determinant and adjoint.

Also Cho [4] studied the generalized inverse of $A \in \mathbb{R}^{n \times n}$ for a regular matrix $A$ and the solution of a fuzzy equation $x \otimes A = b$. Infinite fuzzy relational equations have been discussed by Xue-ping [13] on a complete Brouwerian lattice, such that if $I$ be a finite set and $J$ an infinite index set, $A = (a_{ij})_{I \times J}$ and $B = (b_i)_{i \in I}$, the problem of a minimal solution of a fuzzy relational equation $A \bigodot X = B$, where $\bigodot$ is the sup-inf composite operation, is investigated. Thag [12] discussed the perturb of element $A$ in system $X \otimes A = B$ where $A_{m \times n}$ and $B_{1 \times n}$ are fuzzy matrices and $\otimes$ is max-min composition.

But it happens quite often in modeling real situations using systems of linear equations that the obtained system turns out to be unsolvable. One possibility would be to omit some equations, but the problem of determining the minimum necessary number of those equations is NP-complete, see [11] for systems over a field and [2] for systems over extremal algebras. For fuzzy relational equations (max-min linear systems), Pedrycz [8] proposed a new method by modifying the right-hand side slightly to get a solvable system. Based on Pedrycz’ idea, Cuninghame-Green and Cechlárková [5] devised a polynomial procedure for finding the Chebychev-best approximation by the theory of residuation in fuzzy algebra, using the right-hand side of system. In this paper, we present an efficient algorithm for finding the Chebychev-best approximation using the unknown $x$ of the system.

In the next section we present some definitions. In the section 3 we introduce present two algorithms, first to find $x^*$, principal solution for consistent system and then to find the Chebychev best approximation of $x^*$ for inconsistent system.

2 Definitions and Notations

By a max-min fuzzy algebra $\mathcal{R}$ we mean any linear ordered set $([0,1], \leq, \oplus, \otimes)$ with binary operations of $\oplus = \text{maximum}$ and $\otimes = \text{minimum}$. For any natural $n > 0$, $\mathcal{R}(n)$ denotes set of all $n$-dimensional column vectors over $\mathcal{R}$, and $\mathcal{R}(m,n)$ denotes the set of all matrices of type $m \times n$ over $\mathcal{R}$. For brevity, we shall denote by $M$ and $N$ the index sets $\{1,2,\ldots,m\}$ and $\{1,2,\ldots,n\}$, respectively.

In this paper, a system of linear equations of the form

$$A \otimes x = b \tag{2.1}$$

where the matrix $A \in \mathcal{R}(m,n)$ and the vector $b \in \mathcal{R}(m)$ are given and $x \in \mathcal{R}(n)$ is unknown, is considered. For each system (2.1) the so-called principal solution $x^* = x^*(A,b)$ is defined by

$$x^*(A,b) = A^t \otimes' b, \tag{2.2}$$

where in the $\otimes'$-matrix multiplication the operations between scalars are $\oplus' = \text{minimum}$ and $\otimes'$ is defined by

$$a \otimes' b = \begin{cases} b & a > b, \\ 1 & a \leq b. \end{cases}$$

The inequality $A \otimes x^*(A,b) \leq b$ holds always (see Corollary 2 in [5]), but system (2.1) is solvable if and only if (2.2) is its solution, see [10].
Since we shall change the right-hand side \( b \) and retain the coefficient matrix \( A \), for a given matrix \( A \in \mathbb{R}(m, n) \), let us denote

\[
S(A) = \{ \hat{b} \in \mathbb{R}(m); A \otimes x = \hat{b} \text{ is solvable} \}.
\]

**Definition 2.1.** [5] The Chebychev distance of two vectors \( b, \hat{b} \in \mathbb{R}(m) \) is

\[
\rho(b, \hat{b}) = \max_i |b_i - \hat{b}_i|, \quad i \in M.
\]

The Chebychev distance of a vector \( b \in \mathbb{R}(m) \) and a set \( B \subseteq \mathbb{R}(m) \) is defined as

\[
\rho(b, B) = \inf_{\hat{b} \in B} \rho(b, \hat{b}).
\]

### 3 Algorithms

The main idea of the first algorithm is to evaluate of the principal solution of (2.1) and in second algorithm, the Chebychev best approximation of (2.1) is computed.

**Algorithm 1.** Principal solution.

**Step 0.** Set \( x_1 = x_2 = \cdots = x_n := 1 \) and \( i := 1 \). Arrange the equations of (2.1) by \( b_1 \leq b_2 \leq \cdots \leq b_m \).

**Step 1.** While \((i \leq m \text{ and } N \neq \emptyset)\)

- **For** \( j \in N \),
  - If \( a_{ij} > b_i \) then \( x_j := b_i \) and \( N := N \setminus \{j\} \);
  - \( i := i + 1 \).

The output of Algorithm 1, is the principal solution of (2.1), \( x^* \). The computational complexity of Algorithm 1 is \( O(m \log m + mn) \), but the direct computation of \( x^*(A, b) = A^t \otimes b \) needs \( O(nm^2) \) comparison.

Start from one point of the feasible space of the system (2.1) like \( \bar{x} \) (for example \( \bar{x} = x^* \) where \( x^* \) can be obtained from Algorithm 1) for computing the Chebychev approximation of solution (2.1), when it is unsolvable, by next algorithm. Now we are ready to state the algorithm for a given matrix \( A \in \mathbb{R}(m, n) \) and a vector \( b \in \mathbb{R}(m) \).

**Algorithm 2.** Chebychev approximation.

**Step 0.** Set \( \bar{x} := x^* \) by running of the Algorithm 1.

**Step 1.** If \( A \otimes \bar{x} = b \), then \( \bar{x} \) is the exact solution of (2.1) and **Stop**.

**Step 2.** Choose \( i_0 \in \{1, 2, \ldots, m\} \) such that

\[
\max_i \{ |b_i - \max_{1 \leq k \leq n} \{ \min(a_{ik}, \bar{x}_k) \}| \} := |b_{i_0} - \max_{1 \leq k \leq n} \{ \min(a_{i_0k}, \bar{x}_k) \}|.
\]
If \((\forall j \in N, x_j \geq \min\{a_{ioj}, b_{io}\})\) then \(\bar{x}\) is the Chebychev approximation of solution and Stop.

else choose \(j_0 \in N\) such that

\[
\max_j \{\min(a_{ioj}, b_{io}) - x_j\} := \min(a_{ioj_0}, b_{io}) - x_{j_0} := \text{Gap}.
\]

**Step 3.** Let set \(M_{j_0}\) be equal:

\[
\left\{ i \in M - \{i_0\} \mid \frac{b_i + b_{io}}{2} - x_{j_0} \in [0, \text{Gap}], a_{ij_0} \geq \frac{b_i + b_{io}}{2} \right\};
\]

If \(M_{j_0} \neq\) then set \(x_{j_0} := \min\{a_{ioj_0}, b_{io}\}\) and \(N := N - \{j_0\}\).

else \(\bar{x}\) is the Chebychev approximation of solution and Stop.

else choose \(p \in M_{j_0}\) such that \(\max_i \in M_{j_0} (a_{ij_0} - b_i) := a_{pj_0} - b_{p}\), and set

\[
x_{j_0} := \frac{b_p + b_{ij_0}}{2}, \quad \text{and} \quad N := N - \{j_0\};
\]

If \(N \neq\) go to Step 2,

else \(\bar{x}\) is the Chebychev approximation of solution and Stop.

**Example 1.** [5] Let

\[
\begin{pmatrix}
0.3 & 1 \\
0.8 & 0.3
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
=
\begin{pmatrix}
0.7 \\
0.2
\end{pmatrix}.
\]

From Algorithm 2, we have \(\bar{x} = (0.2 \ 0.2)^t\) and \(A \otimes \bar{x} = (0.2 \ 0.2)^t\), but \(i_0 = 1\) and \(j_0 = 2\), and then \(\bar{x} = \min\{1, 0.7\} = 0.7\) and hence \(\bar{x} = (0.2 \ 0.7)^t\). So it is the Chebychev approximation of solution.

**Example 2.** Let

\[
\begin{pmatrix}
0.2 & 0.7 & 0.4 & 0.3 \\
0.1 & 0.5 & 0.2 & 0.8 \\
0.9 & 0.4 & 0.3 & 0.1 \\
0.3 & 0.7 & 0.5 & 0.3
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix}
=
\begin{pmatrix}
0.3 \\
0.7 \\
0.9 \\
0.2
\end{pmatrix}.
\]

We have \(\bar{x} = (0.2 \ 0.2 \ 0.2 \ 0.2)^t\) and \(i_0 = 3, j_0 = 1, \text{Gap} = 0.7\), then \(\bar{x} = 0.9\) and \(N = \{2, 3, 4\}\). Hence \(A \otimes \bar{x} = (0.2 \ 0.2 \ 0.9 \ 0.3)^t\) and therefore \(i_0 = 2, j_0 = 4, \text{Gap} = 0.5\), hence \(\bar{x} = 0.7\) and \(\bar{x} = (0.9 \ 0.2 \ 0.2 \ 0.7)^t\) with \(A \otimes \bar{x} = (0.3 \ 0.7 \ 0.9 \ 0.3)^t\) and we have the Chebychev best approximation of solution.

**Example 3.** [5] Let

\[
\begin{pmatrix}
0.5 & 0.9 \\
0.6 & 0.3 \\
0.2 & 0.7
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
=
\begin{pmatrix}
0.1 \\
0 \\
0.8
\end{pmatrix}.
\]

Therefore \(\bar{x} = (0 \ 0)^t\) and \(i_0 = 3, j_0 = 2\), and also \(\text{Gap} = 0.7\), \(M_{j_0} = \{1\}\). Then \(p = 1\) and \(x_2 = \frac{0.1 + 0.8}{2}\), hence we have the Chebychev best approximation of solution, \(\bar{x} = (0 \ 0.45)^t\).

Note that the Chebychev best approximation of solution in last example is not unique and \((\alpha \ 0.45)^t\) for \(\alpha \in [0, 0.45]\) is the Chebychev solution too, with \(\text{Gap} = 0.45\).
**Example 4.** Let

\[
\begin{pmatrix}
1 & 0.8 & 0.4 & 0.1 & 0.5 \\
0.3 & 0.5 & 0.2 & 0.8 & 1 \\
0.9 & 0.7 & 0.3 & 0.4 & 0.1 \\
0 & 0.3 & 0.5 & 0.2 & 0.8 \\
0.9 & 0.7 & 0.3 & 0.4 & 0.1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{pmatrix}
= \begin{pmatrix}
0.1 \\
0.7 \\
0.8
\end{pmatrix}.
\]

We have \(\bar{x} = (0.1 \ 0.1 \ 0.7 \ 0.1)^T\) and \(i_0 = 3, \ j_0 = 1, \ \text{Gap} = 0.7\) also \(M_1 = \{1\}\), then \(\bar{x}_1 = (0.1 + 0.8)/2 = 0.45\) and \(N = \{2, 3, 4, 5\}\). Hence \(A \otimes \bar{x} = (0.45 \ 0.7 \ 0.45)^T\) and if choose \(i_0 = 1\) or \(i_0 = 3\) then algorithm will stopped. So we have the Chebychev best approximation of solution.

**Theorem 3.1.** Algorithm 2 returns the Chebychev best approximation of solution (2.1).

**Proof.** First, the equation \(N = N - \{j\}\) in algorithm guaranteed the termination of algorithm and second, if \(y\) is the Chebychev best approximation of solution of (2.1) then there exists \(p \in M\) and \(q \in N\) such that

\[
\rho(Ay, b) = \max_{i \in M} \{|b_i - \max_{j \in N} (a_{ij}, y_j)|\} = |b_p - \max_{j \in N} (a_{pj}, y_j)| = |b_p - \min (a_{pq}, y_q)| = \rho.
\]

If \(a_{pq} \leq y_q\) then \(|b_p - a_{pq}| = \rho\) and \(y_q(\geq a_{pq})\) is arbitrary and also \(a_{pq} < b_p\) (else with increasing \(y_q\), \(\rho\) must be decrease) and then \(y_q = \bar{x}_q\) and implies \(A \otimes \bar{x} = A \otimes y\). If \(a_{pq} > y_q\) then \(|b_p - y_q| = \rho\) and \(y_q \neq b_p\), so we have two cases:

**Case 1.** If \(y_q < b_p\) then \(\bar{x}_q = \min (a_{pq}, b_p) > y_p\) so \(\min (a_{pq}, y_q) < \min (a_{pq}, \bar{x}_q)\) and \(b_p - \min (a_{pq}, y_q) > b_p - \min (a_{pq}, \bar{x}_q)\) which is contradiction.

**Case 2.** If \(y_q > b_p\) then \(\exists i_0 \in M\) such that \(b_{i_0} - \min (a_{i_0q}, y_q) = \rho\) and then

\[
\begin{cases}
y_q - b_p = \rho \\
y_q + b_{i_0} = \rho
\end{cases} \Rightarrow y_q = \frac{b_p + b_{i_0}}{2}.
\]

It says that we have \(\rho(A\bar{x}, b) = \rho(Ay, b) = \rho\). 

\(\square\)

### 4 Conclusions

In this paper, we introduce an efficient algorithm to construct the Chebychev best approximation of solution of (2.1) from the principal solution of (2.1), \(x^*\), within finite steps, at most in \(\min \{m, n\}\) steps. Also to compute \(x^*\) in our algorithm we need fewer comparison.
References


